

Centroid, Area, and Moments of Inertia

Yong-Ming Li

January, 1997

1 Introduction

This design document was originally written for computation of area, centroid, and moments of inertia of lamina (a thin plate of uniform density). It is expanded to compute physical properties of wire and solid.

Area, centroid (or center of mass), and moments of inertia (a measure of an object's resistance to changes in its rotation rate) are usually studied in engineering mechanics. For detailed discussion, one may refer to [Hibbeler'1991].

2 Set of points

Given n points $(\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)$ in space with associated mass m_1, m_2, \dots, m_n , then the position vector of the center of mass is given by:

$$x_c = \frac{1}{M} \sum_{i=1}^n m_i x_i, \quad y_c = \frac{1}{M} \sum_{i=1}^n m_i y_i, \quad z_c = \frac{1}{M} \sum_{i=1}^n m_i z_i,$$

where $M = m_1 + m_2 + \dots + m_n$. If $m_1 = m_2 = \dots = m_n = m$ (uniform mass), then

$$x_c = \frac{1}{n} \sum_{i=1}^n x_i, \quad y_c = \frac{1}{n} \sum_{i=1}^n y_i, \quad z_c = \frac{1}{n} \sum_{i=1}^n z_i.$$

3 Wire

Assume that an open or closed thin wire can be mathematically described by a vector-valued parametric curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ with $t \in [a, b]$ and its density of mass is $\rho(x, y, z)$. Then,

- Length: $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$
- Mass: $M = \int_a^b \rho(x, y, z) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$
- Moment about the y - z plane: $M_{yz} = \int_a^b x(t) \rho(x, y, z) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$
- Moment about the x - z plane: $M_{xz} = \int_a^b y(t) \rho(x, y, z) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$
- Moment about the x - y plane: $M_{xy} = \int_a^b z(t) \rho(x, y, z) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$

The center of mass is given by

$$x_c = \frac{M_{yz}}{M}, \quad y_c = \frac{M_{xz}}{M}, \quad z_c = \frac{M_{xy}}{M}.$$

In most engineering applications, the density of mass is constant so that ρ can be taken out of the integration sign. We may further assume that $\rho = 1$ and hence discard it from our discussion. In this case, the mass M has the same value as the length L though they have different units.

We now consider moments of inertia. To simplify the typing, we denote the small change of arc length by

$$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$$

Since $\mathbf{r}(t)$ is a 3D space curve, the moments of inertia to three axes are

- $I_{xx} = \int_L [y^2(t) + z^2(t)] ds.$
- $I_{yy} = \int_L [x^2(t) + z^2(t)] ds.$
- $I_{zz} = \int_L [x^2(t) + y^2(t)] ds.$

The products of inertia are

- $I_{xy} = I_{yx} = \int_L xy ds.$
- $I_{xz} = I_{zx} = \int_L xz ds.$
- $I_{yz} = I_{zy} = \int_L yz ds.$

We can represent all inertias in a matrix called *the inertia matrix* as follows:

$$\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix},$$

which will be used to determine the principal inertias and axes later.

The inertias we have discussed so far are defined on a coordinate system (\mathbf{O}, x, y, z) where the origin may not coincide with the centroid. Let $(\mathbf{C}, \hat{x}, \hat{y}, \hat{z})$ be a new coordinate system with \mathbf{C} being the centroid; \hat{x} , \hat{y} , and \hat{z} parallel to the x , y , and z axes. Then, the moment of inertia to the x -axis is

$$I_{xx} = \int_L [(y_c + \hat{y})^2 + (z_c + \hat{z})^2] ds = I_{\hat{x}\hat{x}} + L(y_c^2 + z_c^2) + 2y_c \int_L \hat{y} ds + 2z_c \int_L \hat{z} ds,$$

where

$$I_{\hat{x}\hat{x}} = \int_L [\hat{y}^2 + \hat{z}^2] ds$$

is a moment of inertia to the axis \hat{x} passing through the centroid. Noting that \hat{y} and \hat{z} passing through the centroid, we have $\int_L \hat{y} ds = \int_L \hat{z} ds = 0$. Accordingly, $I_{xx} = I_{\hat{x}\hat{x}} + L(y_c^2 + z_c^2)$. We can similarly derive $I_{\hat{y}\hat{y}}$ and $I_{\hat{z}\hat{z}}$. The results are summarized as follows:

$$I_{\hat{x}\hat{x}} = I_{xx} - L(y_c^2 + z_c^2).$$

$$I_{\hat{y}\hat{y}} = I_{yy} - L(x_c^2 + z_c^2).$$

$$I_{\hat{z}\hat{z}} = I_{zz} - L(x_c^2 + y_c^2).$$

The other products of inertia can be calculated in a similar manner and the results written as follows

$$I_{\hat{x}\hat{y}} = I_{xy} - Lx_c y_c.$$

$$I_{\hat{x}\hat{z}} = I_{xz} - Lx_c z_c.$$

$$I_{\hat{y}\hat{z}} = I_{yz} - Ly_c z_c.$$

We now consider the computation of principal moments of inertia and principal axes. It is readily seen that

$$I_{\hat{x}\hat{x}} + I_{\hat{y}\hat{y}} + I_{\hat{z}\hat{z}} = 2 \int_L r^2 ds,$$

where r is a distance to the centroid from an arbitrary point on the curve. It is invariant under any transformation. Therefore, the sum of the moments of inertia is invariant with respect to a coordinate system rotation. In terms of matrix notation, the sum of the moments of inertia is just the sum of the elements on the principal diagonal of the inertia matrix and is known as the trace of that matrix. So the trace of the inertia matrix is unchanged by a coordinate rotation, because the trace of any square matrix is invariant under an orthogonal transformation.

Next the products of inertia are considered. A coordinate rotation of axes can result in a change in the signs of the products of inertia. A 180° rotation about the x axis, for example, reverses the signs of $I_{\hat{x}\hat{y}}$ and $I_{\hat{x}\hat{z}}$, while the sign of $I_{\hat{y}\hat{z}}$ is unchanged. This occurs because the directions of the positive y and z axes are reversed. On the other hand, a 90° rotation about the x axis reverses the sign of $I_{\hat{y}\hat{z}}$. It can be seen that the moments and products of inertia vary smoothly with changes in the orientation of the coordinate system because the direction cosines vary smoothly. Therefore, an orientation can always be found for which a given product of inertia is zero. In other words, the inertia matrix is diagonal. The three mutually orthogonal coordinate axes are known as principal axes in this case, and the corresponding moments of inertia are the principal moments of inertia. The three planes formed by the principal axes are called principal planes.

If I is a principal moment of inertia, then I satisfies the cubic characteristic equation

$$\begin{pmatrix} I_{xx} - I & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} - I & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} - I \end{pmatrix} = 0.$$

This is a eigenvalues and eigenvectors issue. The obtained eigenvalue are the principal moments of inertia and eigenvectors are the principal axes.

4 Generalization

In the previous section, we discussed how to compute physical properties for a wire. All the discussions can be generalized to planar or curved plates and solids. This is done by replacing L by a continuous region D in integral and ds by the infinitesimal mass element dm that may take one of the following values:

- $dm = \rho_V dV$,
- $dm = \rho_A dA$,
- $dm = \rho_l dl$,

where ρ_V , ρ_A , and ρ_l are the volume density, area density and length density.

5 Lamina

5.1 Introduction

A lamina is a thin planar plate with uniform thickness. Mathematically, lamina is a two-dimensional closed planar surface with mass and density. Since it is a planar surface, a closed boundary curve is sufficient for computing its physical properties. Accordingly, computations can be simpler and faster. Therefore, we discuss its physical properties in a different way than that in the previous sections though the end results should be the same if we still stick to the generic approach.

5.2 Definition

Definitions of an area, centroid, and moments of inertia for a lamina can be found in most text books of engineering mechanics (e.g., [Hibbeler'1991]). For completeness, we give a brief review of these definitions in this section.

Let D be a region consisting of the boundary of the lamina and its interior; $\rho(x, y)$ be the density of mass of the lamina. Then,

- Area: $A = \int \int_D dx dy$.
- Mass: $M = \int \int_D \rho(x, y) dx dy$.
- Moment about the x -axis: $M_x = \int \int_D y \rho(x, y) dx dy$.
- Moment about y -axis: $M_y = \int \int_D x \rho(x, y) dx dy$.
- Moment of inertia about the x -axis: $I_{xx} = \int \int_D y^2 \rho(x, y) dx dy$.
- Moment of inertia about the y -axis: $I_{yy} = \int \int_D x^2 \rho(x, y) dx dy$.
- Polar moment of inertia: $I_O = I_{x^2+y^2} = \int \int_D (x^2 + y^2) \rho(x, y) dx dy = I_{xx} + I_{yy}$.
- Product of inertia: $I_{xy} = \int \int_D xy \rho(x, y) dx dy$.

The center of mass (or centroid) is given by

$$x_c = \frac{M_y}{M}, \quad y_c = \frac{M_x}{M}.$$

In most engineering applications, the density of mass of lamina is constant so that ρ can be taken out of the integration sign. We may further assume that $\rho = 1$ by disregarding the constant factor. Therefore, we have the following simpler formulas:

- Mass: $M = \int \int_D dx dy$.
- Moment about the x -axis: $M_x = \int \int_D y dx dy$.
- Moment about y -axis: $M_y = \int \int_D x dx dy$.
- Moment of inertia about the x -axis: $I_{xx} = \int \int_D y^2 dx dy$.
- Moment of inertia about the y -axis: $I_{yy} = \int \int_D x^2 dx dy$.
- Polar moment of inertia: $I_O = I_{xx} + I_{yy}$.

- Product of inertia: $I_{xy} = \int \int_D xy dx dy$.

In engineering mechanics, we often want to know the moments of inertia about the axes passing through the centroid. Let a user defined coordinate system be (O, x, y) and the centroid of a lamina be $C = (x_c, y_c)$. Assume the new coordinate system is (C, ξ, η) with ξ - and η -axis parallel to the x - and y -axis. Then, we have

$$\begin{aligned} I_{xx} &= \int \int y^2 dx dy = \int \int (y_c + \eta)^2 dx dy = y_c^2 \int \int dx dy + 2y_c \int \int \eta dx dy + \int \int \eta^2 dx dy \\ &= y_c^2 A + 2y_c \int \int \eta d\xi d\eta + \int \int \eta^2 d\xi d\eta. \end{aligned}$$

It is noted that the third term represents the moment of inertia about the centroidal axis ξ . The second term is zero since the ξ - and η -axis pass through the centroid, i.e., $\int \int \eta d\xi d\eta = 0$. Thus, we have

$$I_{xx} = y_c^2 A + I_{\xi\xi} \quad \text{or} \quad I_{\xi\xi} = I_{xx} - y_c^2 A.$$

Similarly, we can drive:

- $I_{\xi\eta} = I_{xy} - x_c y_c A$
- $I_{\eta\eta} = I_{yy} - x_c^2 A$
- $I_{\xi^2 + \eta^2} = I_{\xi\xi} + I_{\eta\eta}$.

It should be pointed out that $I_{\xi\eta} = 0$ if a lamina is symmetric to either ξ - or η -axis.

5.3 Principal moments of inertia

Let $I_{\xi\xi}$, $I_{\eta\eta}$, $I_{\xi^2 + \eta^2}$, and $I_{\xi\eta}$ be moments of inertia defined in the coordinate system (C, ξ, η) , where C is the centroid of lamina. We now want to compute moments of inertia about a new system (C, u, v) which is obtained by rotating (C, ξ, η) by θ . It is noted that

$$u = \xi \cos \theta + \eta \sin \theta$$

$$v = \eta \cos \theta - \xi \sin \theta$$

Therefore, we can derive that

- $I_{uu} = \int v^2 dA = I_{\xi\xi} \cos^2 \theta + I_{\eta\eta} \sin^2 \theta - 2I_{\xi\eta} \sin \theta \cos \theta$.
- $I_{vv} = \int u^2 dA = I_{\xi\xi} \sin^2 \theta + I_{\eta\eta} \cos^2 \theta + 2I_{\xi\eta} \sin \theta \cos \theta$.
- $I_{uv} = \int uv dA = I_{\xi\xi} \sin \theta \cos \theta - I_{\eta\eta} \sin \theta \cos \theta + 2I_{\xi\eta}(\cos^2 \theta - \sin^2 \theta)$.
- $I_{u^2 + v^2} = I_{\xi^2 + \eta^2}$.

The above equations may be simplified by using the trigonometric identities

$$\sin(2\theta) = 2 \sin \theta \cos \theta \quad \text{and} \quad \cos(2\theta) = \cos^2 \theta - \sin^2 \theta,$$

in which case

$$\bullet \quad I_{uu} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2} \cos(2\theta) - I_{xy} \sin(2\theta) \quad (1)$$

- $I_{vv} = \frac{I_{xx} + I_{yy}}{2} - \frac{I_{xx} - I_{yy}}{2} \cos(2\theta) + I_{xy} \sin(2\theta)$
- $I_{uv} = \frac{I_{xx} - I_{yy}}{2} \sin(2\theta) + I_{xy} \cos(2\theta)$

Different θ results in different moments of inertia. We now want to determine the θ_p such that I_{uu} and I_{vv} are maximum and minimum. This particular set of axes is called the principal axes, and the corresponding moments of inertia with respect to these axes are called the principal moments of inertia. The θ_p may be found by differentiating, for example, I_{uu} with respect to θ and setting the result equal to zero. Thus,

$$\frac{dI_{uu}}{d\theta} = -2 \frac{I_{\xi\xi} - I_{\eta\eta}}{2} \sin 2\theta - 2I_{\xi\eta} \cos 2\theta = 0.$$

Consequently, at $\theta = \theta_p$, we have

$$\tan 2\theta_p = -\frac{2I_{\xi\eta}}{I_{\xi\xi} - I_{\eta\eta}} \quad (2)$$

and

$$I_{\max}, I_{\min} = \frac{I_{\xi\xi} + I_{\eta\eta}}{2} \pm \sqrt{\left(\frac{I_{\xi\xi} - I_{\eta\eta}}{2}\right)^2 + I_{\xi\eta}^2}.$$

It is noted that equation (2) gives two solutions, θ_1 and θ_2 , which are 90° apart. To find which one is correct, we may substitute θ into equation (1) and check if I_{uu} matches I_{\max} . If not, we should set $\theta = \theta + \pi/2$.

We now consider two principal axes. Let the x - and y -axis be defined respectively by two unit vectors $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Then, the unit vectors that define the principal axes are

$$\mathbf{u} = \mathbf{x} \cos \theta_p + \mathbf{y} \sin \theta_p \quad \text{and} \quad \mathbf{v} = \mathbf{y} \cos \theta_p - \mathbf{x} \sin \theta_p.$$

Replacing θ in (1), we can obtain I_{uu} and I_{vv} which correspond to I_{\min} and I_{\max} .

5.4 Green Theorem

We start our discuss by considering the Green's theorem.

Theorem 5.1 (Green Theorem) *Let C be a piecewise smooth simple closed curve and D be the region consisting of C and its interior. If $P(x, y)$ and $Q(x, y)$ are functions that are continuous and have continuous first partial derivatives throughout an open region R containing D , then*

$$\iint_D \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y} \right) dx dy = \oint_C P dx + Q dy$$

The sign of the right side integration is associated with the orientation of the curve. If we walk along the curve and see the region D on our left side, then the integration is positive. This requirement coincides with the definition of the ACIS about an orientation of a curve. Green's theorem can also be applied to a region D which contains holes, provided we integrated over entire boundary and always keep the region D to the left. This requirement again matches the definition of the ACIS. Hence, Green's theorem will be used to compute properties of lamina which have holes inside.

In CAD systems, curves are often given parametrically as $\mathbf{r}(t) = (x(t), y(t))$ with $t \in [t_0, t_1]$. Let $P(x, y) = -y$ and $Q(x, y) = 0$. From Green theorem, a formula of computing an area for a lamina is given by

$$A = \int \int_D dx dy = \int \int_D \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y} \right) dx dy = - \oint_C y dx = - \int_{t_0}^{t_1} y(t) x'(t) dt.$$

As an example, we compute an area of a circle $x(\theta) = R \cos(\theta)$ and $y(\theta) = R \sin(\theta)$:

$$A = - \int_0^{2\pi} y(\theta) x'(\theta) d\theta = R^2 \int_0^{2\pi} (\sin \theta)^2 d\theta = R^2 \left(\frac{\theta}{2} - \frac{1}{4} \sin 2\theta \right)_0^{2\pi} = R^2 \pi.$$

Similarly, we can derive

$$M_x = \int \int y dx dy = -\frac{1}{2} \oint y^2 dx = -\frac{1}{2} \int_{t_0}^{t_1} y^2(t) x'(t) dt \quad (P = -\frac{1}{2} y^2, Q = 0)$$

$$M_y = \int \int x dx dy = \frac{1}{2} \oint x^2 dy = \frac{1}{2} \int_{t_0}^{t_1} x^2(t) y'(t) dt \quad (P = 0, Q = \frac{1}{2} x^2)$$

$$I_{xx} = \int \int y^2 dx dy = -\frac{1}{3} \oint y^3 dx = -\frac{1}{3} \int_{t_0}^{t_1} y^3(t) x'(t) dt \quad (P = -\frac{1}{3} y^3, Q = 0)$$

$$I_{xy} = \int \int xy dx dy = -\frac{1}{2} \oint xy^2 dx = -\frac{1}{2} \int_{t_0}^{t_1} y^2(t) x(t) x'(t) dt \quad (P = -\frac{1}{2} xy^2, Q = 0)$$

$$I_{yy} = \int \int x^2 dx dy = \frac{1}{3} \oint x^3 dy = \frac{1}{3} \int_{t_0}^{t_1} x^3(t) y'(t) dt \quad (P = 0, Q = \frac{1}{3} x^3)$$

5.5 Gauss-Legendre integration

It is seen that computation of an area and moments is actually a requirement of computing the integration of the type:

$$\int_a^b f(t) dt.$$

To compute the above integration numerically, the well-known method is the Gauss-Legendre integration rule which has the following form:

$$\int_a^b f(t) dt \simeq \sum_{i=0}^n A_i f(x_i)$$

The above integration rule is exact if $f(x)$ is a polynomial of degree not more than $2n + 1$. In our case, $f(t)$ is either polynomial or rational function. If it is former case, $f(t)$ is a polynomial and thus the integration rule is exact. If it is the latter case, $f(t)$ can be approximated by a polynomial function since $f(t)$ is an analytic function. Therefore, we can usually obtain accurate results very efficiently using the Gauss-Legendre integration rule.

5.6 Applying Green's theorem to a composite curve

It is noted that Green's theorem do not require C to be a single closed curve. Instead, C could be a piecewise closed curve. If C is made of piecewise curves C_i (C_i may be lines, arcs, and B-spline curves), then the Green's formula becomes

$$\oint_C P(x, y) dx + Q(x, y) dy = \sum \int_{C_i} P(x, y) dx + Q(x, y) dy.$$

As an extreme case where C_i are all lines defined by points (x_i, y_i) and (x_{i+1}, y_{i+1}) , then the Green formula is given by

$$\oint_C P(x, y)dx + Q(x, y)dy = \sum \int_{x_i}^{x_{i+1}} [P(x, y) + Q(x, y)y'(x)]dx.$$

We now consider area, moments, and moments of inertia. For the purpose of simplifying notation, let any two adjacent points be denoted by (x_1, y_1) and (x_2, y_2) . Then, a line joining the two points is given by

$$y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) = y_1 + \lambda(x - x_1).$$

Referring to previous sections, the area of lamina is given by

$$A = - \oint ydx = - \sum \int_{x_1}^{x_2} ydx = -\frac{1}{2} \sum (y_1 + y_2)(x_2 - x_1)$$

We note that $(y_1 + y_2)(x_2 - x_1)/2$ is the area of trapezoid. The moments of lamina are given by

$$M_x = -\frac{1}{2} \oint y^2 dx = -\frac{1}{2} \sum \int_{x_1}^{x_2} [y_1 + \lambda(x - x_1)]^2 dx = -\frac{1}{6} \sum (x_2 - x_1)(y_1^2 + y_1 y_2 + y_2^2)$$

$$M_y = \frac{1}{2} \oint x^2 dy = \frac{1}{2} \sum \int_{y_1}^{y_2} [x_1 + \frac{1}{\lambda}(y - y_1)]^2 dy = \frac{1}{6} \sum (y_2 - y_1)(x_1^2 + x_1 x_2 + x_2^2)$$

The moments of inertia are given by

$$I_{xx} = -\frac{1}{3} \oint y^3 dx = -\frac{1}{3} \sum \int_{x_1}^{x_2} [y_1 + \lambda(x - x_1)]^3 dx = -\frac{1}{12} \sum (x_2 - x_1)(y_1^3 + y_1^2 y_2 + y_1 y_2^2 + y_2^3)$$

$$I_{yy} = \frac{1}{3} \oint x^3 dy = \frac{1}{3} \sum \int_{y_1}^{y_2} [x_1 + \frac{1}{\lambda}(y - y_1)]^3 dy = \frac{1}{12} \sum (y_2 - y_1)(x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3)$$

$$\begin{aligned} I_{xy} &= -\frac{1}{2} \oint xy^2 dx = -\frac{1}{2} \sum \int_{x_1}^{x_2} x[y_1 + \lambda(x - x_1)]^2 dx \\ &= -\frac{1}{6\lambda} \sum \left\{ x[y_1 + \lambda(x - x_1)]^3 - \int_{x_1}^{x_2} [y_1 + \lambda(x - x_1)]^3 dx \right\} \\ &= -\frac{1}{6\lambda} \left[x_2 y_2^3 - x_1 y_1^3 - \frac{1}{4}(x_2 - x_1)(y_2^3 + y_2^2 y_1 + y_2 y_1^2 + y_1^3) \right] \end{aligned}$$

It is noted that if $y_1 = y_2$, then $\frac{1}{\lambda} = \infty$. In this case, we represent the line parametrically as

$$y = y_1 \quad \text{and} \quad x = x_1 + (x_2 - x_1)t, \quad t \in [0, 1].$$

Thus,

$$I_{xy} = -\frac{1}{2} \oint xy^2 dx = -\frac{1}{4} y_1^2 (x_2^2 - x_1^2).$$

5.7 Ellipse: a special case

Let \mathbf{u} and \mathbf{v} denote the unit vectors of the major and minor axes of an ellipse, and a, b the length of major and minor axes respectively. Then, we may write an ellipse parametrically as

$$u(\theta) = a \cos \theta \quad \text{and} \quad v(\theta) = b \sin \theta.$$

Let ϕ be the angle measured anti-clock-wisely from the x -axis to the major axis \mathbf{u} . Then, the x and y coordinates are given by

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} u(\theta) \\ v(\theta) \end{pmatrix} = \begin{pmatrix} a \cos \phi \cos \theta - b \sin \phi \sin \theta \\ a \sin \phi \cos \theta + b \cos \phi \sin \theta \end{pmatrix}.$$

In calculus, we have learned that

- $\oint \cos^2 \theta d\theta = \oint \sin^2 \theta d\theta = \pi$,
- $\oint \cos^3 \theta d\theta = \oint \sin^3 \theta d\theta = 0$,
- $\oint \sin^2 \theta \cos^2 \theta d\theta = \pi/4$,
- $\oint \cos^4 \theta d\theta = \oint \sin^4 \theta d\theta = 3\pi/4$,
- $\oint \cos^3 \theta \sin \theta d\theta = \oint \sin^3 \theta \cos \theta d\theta = 0$.

Let $a_s = a \sin \phi$, $a_c = a \cos \phi$, $b_s = b \sin \phi$, and $b_c = b \cos \phi$. Then, $x(\theta) = a_c \cos \theta - b_s \sin \theta$ and $y(\theta) = a_s \cos \theta + b_c \sin \theta$. Accordingly, we can derive

- $I_{xx} = -\frac{1}{3} \oint y^3(\theta) dx(\theta) = \frac{\pi}{4} (a_c b_c + a_s b_s) (a_s^2 + b_c^2)$,
- $I_{xy} = -\frac{1}{2} \oint y^2(\theta) x(\theta) dx(\theta) = \frac{\pi}{4} (a_c b_c + a_s b_s) (a_c a_s - b_c b_s)$,
- $I_{yy} = \frac{1}{3} \oint x^3(\theta) dy(\theta) = \frac{\pi}{4} (a_c b_c + a_s b_s) (a_c^2 + b_s^2)$.

5.8 General organization

In applications, a lamina may be defined by a curve made of lines, arcs, and B-spline curves. For efficient purpose, we may compute line/curve integrals of area and moments along lines and curves. Due to the additive property of areas and moments, we can obtain the total area and moments of lamina by simply add individuals together. To do so, it is however important to ensure that the orientation of the curve is same.

It is easy to implement line integrals of area and moments. For a B-spline curve, the procedures are summarized below:

1. Convert B-spline curve into Bézier curves.
2. With respect to each Bézier curve, compute signed area A_i , moments (M_x and M_y) and moments of inertia (I_{xx} , I_{xy} , I_{yy}).
3. Compute $\sum A_i$, $\sum M_x$, $\sum M_y$, etc..

References

- [1] Hibbeler, R.C., *Engineering Mechanics: Statics*, Prentice Hall, 1991.